Thermodynamics of Rotating Black Branes in Gauss-Bonnet-nonlinear Maxwell Gravity

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Abstract

We consider the Gauss-Bonnet gravity in the presence of a new class of nonlinear electromagnetic field, namely, power Maxwell invariant. By use of a suitable transformation, we obtain a class of real rotating solutions with k rotation parameters and investigate some properties of the solutions such as existence of singularity(ies) and asymptotic behavior of them. Also, we calculate the finite action, thermodynamic and conserved quantities of the solutions and using the the Smarr-type formula to check the first law of thermodynamics.

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In recent years, one of the great developments in general relativity was the discovery of a close relationship between black hole mechanics and the ordinary laws of thermodynamics. The existence of this close relationship between these laws may provide us with a key to our understanding of the fundamental nature of black holes as well as to our understanding of some aspects of the nature of thermodynamics itself. Also, it is notable that the black hole is an object that is considered in classical and quantum point of view and so one hopes to gain insight into the nature of quantum gravity by studying the thermodynamics of black holes.

The laws of black hole mechanics are analogous to the laws of thermodynamics. The quantities of particular interest in gravitational thermodynamics are the physical entropy S and the temperature β^{-1}, where these quantities are respectively proportional to the area and surface gravity of the event horizon [1]. Other black hole properties, such as energy, angular momentum and conserved charges can also be given a thermodynamic interpretation. In finding the thermodynamic quantities, one should use the quasilocal definitions for the thermodynamic variables. By quasilocal, we mean that the quantity is constructed from information that exists on the boundary of a gravitating system alone. Just as the Gauss law, such quasilocal quantities will yield information about the spacetime contained within the system boundary. One of the advantage of using such a quasilocal method is that the formalism does not depend on the particular asymptotic behavior of the system, so one can accommodate a wide class of spacetimes with the same formalism.

In this paper, we attempt to construct the rotating black brane solutions of Gauss-Bonnet gravity in the presence of a nonlinear Maxwell field, namely, power Maxwell invariant, and investigate their thermodynamics properties. In what follows, at first, we present some considerable works on higher derivative gravity as well as power Maxwell invariant theory.

On one hand, since the field equations of gravity are generally covariant and of second order derivatives in the metric tensor, one would naively expect these equations to be derived from an action principle involving metric tensor and its first and second order derivatives [2], analogous to the situation for many other field theories of physics. In recent years, there have been considerable works for understanding the role of the higher curvature terms from various points of view, especially with regard to higher dimensional black hole physics. For example, thermodynamics and other properties of the static black hole solutions in Gauss-Bonnet gravity have been found by many authors [3–9] Also, the Taub-NUT/bolt solutions

of higher derivative gravity and their thermodynamics properties have been constructed [10–13]. Not long ago, M. H. Dehghani introduced two new classes of rotating solutions of second order Lovelock gravity and investigated their thermodynamics [14, 15].

On the other hand, in recent years there has been aroused interest about the solutions whose source is Maxwell invariant raised to the power s, i.e., $(F^{\mu\nu}F_{\mu\nu})^s$ as the source of geometry in Einstein and higher derivative gravity [16]. This theory is considerably richer than that of the linear electromagnetic field and in the special case ($s = 1$) it can reduces to linear field. Also, it is valuable to find and analyze the effects of exponent s on the behavior of the new solutions and the laws of black hole mechanics [17]. In addition, in higher dimensional gravity, for the special choice $s = d/4$, where $d =$ dimension of the spacetime is a multiple of 4, it yields a traceless Maxwell's energy-momentum tensor which leads to conformal invariance [18]. In Ref. [19], higher dimensional, direct analogues of the usual $d = 4$ Einstein–Yang-Mills gravity have been studied.

In the rest of the paper, we give a brief definition of the field equations of Gauss-Bonnet gravity in the presence of nonlinear electromagnetic field and present a new class of rotating black brane solutions and investigate their properties.

Here we present the Gauss-Bonnet gravity, which contains the first three terms of Lovelock gravity in the presence of nonlinear electromagnetic field. The action is

$$\begin{aligned} I_G &= -\frac{1}{16\pi}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\left\{R-2\Lambda+\alpha L_{GB}+\kappa L(F)\right\} \\ &\quad -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{n}x\sqrt{-\gamma}\left\{K+2\alpha\left(J-2\widehat{G}_{ab}K^{ab}\right)\right\}, \end{aligned} \tag{1}$$

where R is the Ricci scalar, Λ is the cosmological constant, α is the Gauss-Bonnet coefficient with dimension $(\text{length})^2$and L_{GB} is the Lagrangian of Gauss-Bonnet gravity

$$L_{GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

where $R_{\mu\nu}$ and $R_{\mu\nu\gamma\delta}$ are Ricci and Riemann tensors of the manifold $\mathcal{M}$, κ is a constant in which one can set it to avoid of negative energy density and present well-defined solutions and $L(F)$ is the Lagrangian of power Maxwell invariant theory

$$L(F) = -F^s. \tag{2}$$

In Eq. (2), $F = F^{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is electromagnetic tensor field and A_μ is the vector potential. In the limit $s = 1$, $L(F)$ reduces to the standard Maxwell form

$L(F) = -F$. The second integral in Eq. (1) is a boundary term which is chosen such that the variational principle is well defined [20]. In this term, γ_{ab} is induced metric on the boundary $\partial\mathcal{M}$, K is trace of extrinsic curvature K^{ab} of the boundary, $\widehat{G}^{ab}(\gamma)$ is Einstein tensor calculated on the boundary, and J is trace of

$$J_{ab} = \frac{1}{3}(K_{cd}K^{cd}K_{ab} + 2KK_{ac}K_b^c - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}). \tag{3}$$

Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$ and electromagnetic field A_μ, the equations of gravitation and electromagnetic fields are obtained as

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{\alpha}{2}\big(8R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 4R_\mu^{\ \rho\sigma\lambda}R_{\nu\rho\sigma\lambda} - 4RR_{\mu\nu} + \\ 8R_{\mu\lambda}R^\lambda{}_\nu + g_{\mu\nu}L_{GB}\big) = 2\kappa\left(sF_{\mu\rho}F_\nu{}^\rho F^{s-1} - \frac{1}{4}g_{\mu\nu}F^s\right), \end{aligned} \tag{4}$$

$$\partial_\mu\left(\sqrt{-g}F^{\mu\nu}F^{s-1}\right) = 0, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor.

Equation (4) does not contain the derivative of the curvatures and therefore the derivatives of the metric higher than two do not appear. Thus, the Gauss-Bonnet gravity is a special case of higher derivative gravity. Hereafter we consider the metric of $(n+1)$-dimensional rotating spacetime with k rotation parameters in the form [21, 22]

$$\begin{aligned} ds^2 &= -f(\rho)\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi_i\right)^2 + \frac{\rho^2}{l^4}\sum_{i=1}^{k}\left(a_i dt - \Xi l^2 d\phi_i\right)^2 \\ &\quad + \frac{d\rho^2}{f(\rho)} - \frac{\rho^2}{l^2}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + \rho^2 dX^2, \end{aligned} \tag{6}$$

where $\Xi = \sqrt{1 + \sum_i^k a_i^2/l^2}$ and dX^2 is the Euclidean metric on the $(n-1-k)$-dimensional submanifold. The rotation group in $(n+1)$ dimensions is $SO(n)$ and therefore $k \leq [n/2]$. Using the gauge potential ansatz

$$A_\mu = h(\rho)\left(\Xi\delta_\mu^0 - \delta_\mu^i a_i\right) \text{(no sum on } i) \tag{7}$$

and solving Eq. (5), we obtain

$$h(\rho) = \begin{cases} C, & s = 0, \frac{1}{2} \\ C\ln(\rho), & s = \frac{n}{2} \\ C\rho^{(2s-n)/(2s-1)}, & \text{otherwise} \end{cases}, \tag{8}$$

where C is an integration constant which is related to the charge parameter. One may note that as $s = 1$, A_μ of Eq. (7) reduces to the gauge potential of the linear Maxwell field [14] as it should be. To find the function $f(\rho)$, one may use any components of Eq. (4). After some calculations, one can show that the solution of field equation (4), can be written as

$$f(\rho) = \frac{2\rho^2}{(n-1)\gamma}\left(1 - \sqrt{1 + \frac{2\gamma\Lambda}{n} + \frac{\gamma m}{\rho^n} - \kappa\gamma\Gamma(\rho)}\right), \tag{9}$$

where

$$\Gamma(\rho) = \begin{cases} 0, & s = 0, \frac{1}{2} \\ 2^{n/2}(n-1)C^n \frac{\ln(\rho)}{\rho^n}, & s = \frac{n}{2} \\ \frac{(2s-1)^2}{2s-n}\left(\frac{-2C^2\rho^{-2(n-1)/(2s-1)}(2s-n)^2}{(2s-1)^2}\right)^s, & \text{Otherwise} \end{cases}, \tag{10}$$

$$\gamma = \frac{4\alpha(n-2)(n-3)}{(n-1)}. \tag{11}$$

The metric function $f(\rho)$ for the uncharged solution ($C = 0$) is real in the whole range $0 \leq \rho < \infty$ provided $\alpha \leq -n/(8\gamma\Lambda)$, but for charged solution it is real only in the range $r_0 \leq \rho < \infty$ where r_0 is the largest real root of the following equation

$$nr_0^n + \gamma\left(2\Lambda r_0^n + nm - \kappa n r_0^n \Gamma_0\right) = 0, \tag{12}$$

where $\Gamma_0 = \Gamma(\rho = r_0)$. In order to restrict the spacetime to the region $\rho \geq r_0$, we introduce a new radial coordinate r as

$$r = \sqrt{\rho^2 - r_0^2} \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_0^2}dr^2. \tag{13}$$

With this new coordinate, the above metric (6) becomes

$$\begin{aligned} ds^2 = & -f(r)\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi_i\right)^2 + \frac{(r^2 + r_0^2)}{l^4}\sum_{i=1}^{k}\left(a_i dt - \Xi l^2 d\phi_i\right)^2 \\ & -\frac{r^2 + r_0^2}{l^2}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 dr^2}{(r^2 + r_0^2)f(r)} + (r^2 + r_0^2)dX^2, \end{aligned} \tag{14}$$

where now the functions $h(\rho)$ and $f(\rho)$ change to

$$h(r) = \begin{cases} C, & s = 0, \frac{1}{2} \\ \frac{1}{2}C\ln(r^2 + r_0^2), & s = \frac{n}{2} \\ C(r^2 + r_0^2)^{(2s-n)/[2(2s-1)]}, & \text{otherwise} \end{cases}, \tag{15}$$

$$f(r)=\frac{2\left(r^2+r_0^2\right)}{(n-1)\gamma}\left(1-\sqrt{1+\frac{2\gamma\Lambda}{n}+\frac{\gamma m}{(r^2+r_0^2)^{n/2}}-\kappa\gamma\Gamma(r)}\right), \tag{16}$$

$$\Gamma(r) = \left\{\begin{array}{cc} 0, & s=0,\frac{1}{2} \\ 2^{n/2}(n-1)C^n\frac{\ln(r^2+r_0^2)}{2(r^2+r_0^2)^{n/2}}, & s=\frac{n}{2} \\ \frac{(2s-1)^2}{2s-n}\left(\frac{-2C^2(2s-n)^2}{(2s-1)^2(r^2+r_0^2)^{(n-1)/(2s-1)}}\right)^s, & \text{Otherwise} \end{array}\right., \tag{17}$$

These solutions reduce to the solutions presented in Ref. [14] as $s=1$, to those of Ref. [23] as α vanishes and to the solutions given in Ref. [24] as α goes to zero and $s=1$, simultaneously.

Properties of the solutions

In order to study the general structure of these spacetime, we first look for the essential singularity(ies). After some algebraic manipulation, one can show that for the rotating metric (14), the Kretschmann scalar is

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= \frac{(r^2+r_0^2)^2 f''^2(r)}{r^4}-\frac{2r_0^2(r^2+r_0^2)f''(r)f'(r)}{r^5}+ \\ &\frac{[2(n-1)r^4+r_0^4)]f'^2(r)}{r^6}+\frac{2(n-1)(n-2)f^2(r)}{(r^2+r_0^2)^2}, \end{aligned} \tag{18}$$

where prime and double prime are first and second derivative with respect to r , respectively. It is straightforward to show that the Kretschmann scalar (18) with metric function (16) diverges at $r=0$ and is finite for $r\neq 0$. Also one can show that other curvature invariants (such as Ricci square, Weyl square and so on) diverges at $r=0$. Thus, there is a curvature singularity located at $r=0$.

In Einstein gravity coupled to power Maxwell invariant [23], it is shown that for $s>n/2$ and $s<0$, the singularity at $r=0$ for the solutions with non-negative mass is spacelike, and therefore it is unavoidable. These solutions with positive mass present black branes with one horizon. But here, in Gauss-Bonnet gravity, we have a extra parameter, α, and one can set the parameters of the solutions to have timelike singularity for all values of nonlinear parameter.

Second, we investigate the effects of the nonlinearity on the asymptotic behavior of the solutions. It is worthwhile to mention that for $0<s<\frac{1}{2}$, the asymptotic dominant term of Eq. (16) is charge term, $\Gamma(r)$, and the presented solutions are not asymptotically AdS, but for the cases $s<0$ or $s>\frac{1}{2}$ (include of $s=\frac{n}{2}$), the asymptotic behavior of rotating Einstein-nonlinear Maxwell field solutions are the same as linear AdS case. It is easy to show

that the electromagnetic field is zero for the cases $s = 0, 1/2$, and the metric function (16) does not possess a charge term ($\Gamma(r) = 0$) and it corresponds to uncharged asymptotically AdS one.

The metric (14) has two types of Killing and event horizons. The Killing horizon is a null surface whose null generators are tangent to a Killing field. It is proved that a stationary black hole event horizon should be a Killing horizon in the four-dimensional Einstein gravity [25]. This proof can not obviously be generalized to higher order gravity, but the result is true for all the known static solutions. Although our solution is not static, the Killing vector,

$$\chi = \partial_t + \sum_i^k \Omega_i \partial_{\phi_i}, \tag{19}$$

is the null generator of the event horizon, where Ω_i is the ith component of angular velocity of the outer horizon. The angular velocities Ω_i's may be obtained by analytic continuation of the metric. Setting $a_i \to i a_i$ yields the Euclidean section of (14), whose regularity at $r = r_+$ (requiring the absence of conical singularity at the horizon in the Euclidean sector of the black brane solutions) requires that we should identify $\phi_i \sim \phi_i + \beta\Omega_i$. One obtains

$$\Omega_i = \frac{a_i}{\Xi l^2}. \tag{20}$$

The temperature may be obtained through the use of regularity at $r = r_+$ or definition of surface gravity,

$$T_+ = \beta^{-1} = \frac{1}{2\pi}\sqrt{-\frac{1}{2}\left(\nabla_\mu \chi_\nu\right)\left(\nabla^\mu \chi^\nu\right)}, \tag{21}$$

where χ is the Killing vector (19). It is straightforward to show that

$$T_+ = \frac{f'(r_+)}{4\pi\Xi}\sqrt{1 + \frac{r_0^2}{r_+^2}} = -\frac{2\Lambda + (n-1)\kappa\Upsilon}{4\pi\Xi(n-1)}\sqrt{r_+^2 + r_0^2}, \tag{22}$$

$$\Upsilon = \begin{cases} 0, & s = 0, \frac{1}{2} \\ 2^{n/2} C^n (r_+^2 + r_0^2)^{-n/2}, & s = \frac{n}{2} \\ \frac{(2s-1)}{n-1}\left(\frac{-2C^2(2s-n)^2}{(2s-1)^2 (r_+^2 + r_0^2)^{(n-1)/(2s-1)}}\right)^s, & \text{otherwise} \end{cases}. \tag{23}$$

Using the fact that the temperature of the extreme black brane is zero, it is easy to show that the condition for having an extreme black hole is that the mass parameter is equal to m_{ext}, where m_{ext} is given as

$$m_{\text{ext}} = \zeta - \frac{8\Lambda(r^2 + r_0^2)^{n/2}}{n(n-4)} + \frac{(r^2 + r_0^2)^{n/2}\left[\sqrt{(n-1)(n-2)\xi} - (n-1)(n-2)\right]}{\alpha(n-2)(n-3)(n-4)^2} \tag{24}$$

where

$$\chi = \left\{ \begin{array}{cc} 2^{n/2}\kappa(n-1)C^{n}(r^{2}+r_{0}^{2})^{-n/2}, & s=\frac{n}{2} \\ (2s-1)\kappa\left(\frac{-2C^{2}(n-2s)^{2}}{(2s-1)^{2}(r^{2}+r_{0}^{2})^{(n-1)/(2s-1)}}\right)^{s}, & \text{otherwise} \end{array} \right. ,$$

$$\zeta = \left\{ \begin{array}{cc} \frac{2^{n/2}\kappa(n-1)C^{n}[(n-4)\ln(r^{2}+r_{0}^{2})-2]}{2(n-4)}, & s=\frac{n}{2} \\ -\frac{2\kappa(2s-1)[s(n-5)+2](r^{2}+r_{0}^{2})^{n/2}}{(n-4)(n-2s)}\left(\frac{-2C^{2}(n-2s)^{2}}{(2s-1)^{2}(r^{2}+r_{0}^{2})^{(n-1)/(2s-1)}}\right)^{s}, & \text{otherwise} \end{array} \right. ,$$

$$\xi = (n-1)(n-2)+8\alpha\Lambda(n-3)(n-4)+4(n-3)(n-4)\alpha\chi.$$

The metric of Eqs. (14), (16) and (17) presents a black brane solution with inner and outer horizons, provided the mass parameter m is greater than m_{ext}, an extreme black brane for $m=m_{\text{ext}}$ and a naked singularity otherwise.

Here we, first, calculate the thermodynamic and conserved quantities of the black brane. Second, we obtain a Smarr-type formula for the mass as a function of the entropy, the angular momentum and the charge of the solution and finally check the first law of thermodynamics.

Denoting the volume of the hypersurface at $r=$constant and $t=$constant by V_{n-1}, the charge of the black brane, Q, can be found by calculating the flux of the electromagnetic field at infinity, yielding

$$Q = \left\{ \begin{array}{cc} \frac{V_{n-1}(-1)^{(n+2)/2}2^{n/2}\Xi C^{n-1}}{8\pi}, & s=\frac{n}{2} \\ \frac{V_{n-1}(-1)^{s+1}2^{s}\Xi}{8\pi}\left(\frac{(2s-n)C}{(2s-1)}\right)^{2s-1}, & \text{otherwise} \end{array} \right. \tag{25}$$

The electric potential Φ, measured at infinity with respect to the horizon, is defined by [26]

$$\Phi = A_{\mu}\chi^{\mu}\left|_{r\to\infty} - A_{\mu}\chi^{\mu}\right|_{r=r_{+}}, \tag{26}$$

where χ is the null generator of the horizon given by Eq. (19). One finds

$$\Phi = \frac{-C}{\Xi}\left\{ \begin{array}{cc} 1, & s=0,\frac{1}{2} \\ \frac{1}{2}\ln(r_{+}^{2}+r_{0}^{2}), & s=\frac{n}{2} \\ (r_{+}^{2}+r_{0}^{2})^{(2s-n)/[2(2s-1)]}, & \text{otherwise} \end{array} \right. . \tag{27}$$

Black hole entropy typically satisfies the so-called area law, which states that the entropy of a black hole equals one-quarter of the area of its horizon. This near universal law applies to almost all kinds of black objects in Einstein gravity [27]. However in higher derivative gravity the area law is not satisfied in general [28]. For asymptotically flat black hole solutions of Lovelock gravity, the entropy may be written as [29]

$$S = \frac{1}{4}\sum_{k=1}^{[(d-1)/2]} k\alpha_{k}\int d^{n-1}x\sqrt{\tilde{g}}\tilde{\mathcal{L}}_{k-1}, \tag{28}$$

where the integration is done on the $(n-1)$-dimensional spacelike hypersurface of Killing horizon, $\tilde{g}_{\mu\nu}$ is the induced metric on it, $\tilde{g}$ is the determinant of $\tilde{g}_{\mu\nu}$ and $\tilde{\mathcal{L}}_k$ is the kth order Lovelock Lagrangian of $\tilde{g}_{\mu\nu}$. The asymptotic behavior of the black branes we are considering is not flat, and therefore we calculate the entropy through the use of Gibbs-Duhem relation

$$S=\frac{1}{T}(\mathcal{M}-\Gamma_i\mathcal{C}_i)-I, \tag{29}$$

where I is the finite total action evaluated on the classical solution, and $\mathcal{C}_i$ and Γ_i are the conserved charges and their associate chemical potentials respectively.

In general the action I_G, is divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. A systematic method of dealing with this divergence in Einstein gravity is through the use of the counterterms method inspired by the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [30]. This conjecture, which relates the low energy limit of string theory in asymptotically anti de-Sitter spacetime and the quantum field theory living on the boundary of it, have attracted a great deal of attention in recent years. This equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computation on the other side. A dictionary translating between different quantities in the bulk gravity theory and their counterparts on the boundary has emerged, including the partition functions of both theories. This conjecture is now a fundamental concept that furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime [31]. It has also been applied to the case of black holes with constant negative or zero curvature horizons [24] and rotating higher genus black branes [32]. Although the AdS/CFT correspondence applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [33].

All of the work mention in the last paragraph was limited to Einstein gravity. Although the counterterms in Lovelock gravity should be a scalar constructed from Riemann tensor as in the case of Einstein gravity, they are not known for the case of Lovelock gravity till now. But, for the solutions with flat boundary, $\widehat{R}_{abcd}(\gamma)=0$, there exists only one boundary counterterm

$$I_{\rm ct}=-\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^nx\sqrt{-\gamma}\left(\frac{n-1}{l_{\rm eff}}\right), \tag{30}$$

where $l_{\rm eff}$ is a scale length factor that depends on l and α, that must reduce to l as α goes

to zero. One may note that this counterterm has exactly the same form as the counterterm in Einstein gravity for a spacetime with zero curvature boundary in which l is replaced by l_{eff}.

Using Eqs. (1) and (30), the finite action can be calculated as

$$I = \frac{V_{n-1}\Xi l^2(r_+^2+r_0^2)^{(n-1)/2}}{4}\left(\Pi-\frac{1}{l^2}\right), \tag{31}$$

$$\Pi = \left\{\begin{array}{cc} \frac{(n-1)m}{n}(r_+^2+r_0^2)^{-n/2}, & s=0,\frac{1}{2} \\ \frac{(n-1)m+(-1)^{(n+2)/2}2^{n/2}C^n\ln(r_+^2+r_0^2)}{n-2^{n/2}l^2\kappa C^n}, & s=\frac{n}{2} \\ \frac{(n-1)^2(2s-1)^{2s-1}m(r_+^2+r_0^2)^{(n-2s)/[2(2s-1)]}-(-1)^s2^{s+1}(n-1)(2s-n)^{2s-1}C^{2s}}{n(n-1)(2s-1)^{2s-1}-(-1)^s2^sl^2\kappa(2s-n)^{2s}C^{2s}}, & \text{otherwise} \end{array}\right. , \tag{32}$$

Having the total finite action, I_G+I_{ct}, one can use the Brown-York definition of stress energy tensor [34] to construct a divergence-free stress energy tensor. For the case of manifolds with zero curvature boundary, the finite stress energy tensor is [35, 36]

$$T^{ab} = \frac{1}{8\pi}\{(K^{ab}-K\gamma^{ab})+2\alpha(3J^{ab}-J\gamma^{ab})-\left(\frac{n-1}{l_{\mathrm{eff}}}\right)\gamma^{ab}\} \tag{33}$$

One may note that when α goes to zero, the stress energy tensor (33) reduces to that of Einstein gravity. To compute the conserved charges of the spacetime, we choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM form

$$\gamma_{ab}dx^a dx^a = -N^2dt^2+\sigma_{ij}\left(d\varphi^i+V^idt\right)\left(d\varphi^j+V^jdt\right), \tag{34}$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress energy tensors of Eq. (33) can be written as

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{35}$$

where σ is the determinant of the metric σ_{ij}, and n^a is the timelike unit normal vector to the boundary $\mathcal{B}$. For boundaries with timelike ($\xi=\partial/\partial t$) and rotational ($\varsigma=\partial/\partial\varphi$) Killing vector fields, one obtains the quasilocal mass and angular momentum

$$M = \int_{\mathcal{B}} d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{36}$$

$$J = \int_{\mathcal{B}} d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\varsigma^b, \tag{37}$$

provided the surface $\mathcal{B}$ contains the orbits of ς. These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary $\mathcal{B}$. Using Eqs. (36) and (37), the mass and angular momenta of the solution are calculated as

$$\begin{aligned} M &= \frac{V_{n-1}}{16\pi} m \left(n\Xi^2 - 1 \right), \qquad (38) \\ J_i &= \frac{V_{n-1}}{16\pi} n\Xi m a_i. \qquad (39) \end{aligned}$$

Now using Gibbs-Duhem relation (29) and Eqs. (25) - (39) and (31), one obtains

$$S = \frac{V_{n-1}\Xi}{4}(r_+^2 + r_0^2)^{(n-1)/2}. \qquad (40)$$

This shows that the entropy obeys the area law for our case where the horizon curvature is zero.

Calculating all the thermodynamic and conserved quantities of the black brane solutions, we now check the first law of thermodynamics for our solutions. We obtain the mass as a function of the extensive quantities S, $\mathbf{J}$, and Q. Using the expression for the mass, the angular momenta, the charge and the entropy given in Eqs. (25), (38), (39), (40) and the fact that $f(r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S, \mathbf{J}, Q) = \frac{(nZ-1)J}{nl\sqrt{Z(Z-1)}}, \qquad (41)$$

where $J = |\mathbf{J}| = \sqrt{\sum_i^k J_i^2}$ and $Z = \Xi^2$ is the positive real root of the following equation

$$2\Lambda + \frac{16\pi J}{lZ\sqrt{Z^2-1}} \left(\frac{Z}{4S} \right)^{n/(n-1)} - \kappa n \Psi = 0, \qquad (42)$$

$$\Psi = \left\{ \begin{array}{cc} 0, & s = 0, \frac{1}{2} \\ 2^{n/[2(n-1)]} \left[\frac{(-1)^{(n+2)/2}\pi Q}{S} \right]^{n/(n-1)} \ln\left(\frac{4S}{Z}\right), & s = \frac{n}{2} \\ \frac{(-1)^s 2^{s/(2s-1)}(2s-1)^2}{2s-n} \left(\frac{(-1)^{s+1}\pi Q}{S} \right)^{2s/(2s-1)}, & \text{Otherwise} \end{array} \right. .$$

One may then regard the parameters S, J_i's, and Q as a complete set of extensive parameters for the mass $M(S, \mathbf{J}, Q)$ and define the intensive parameters conjugate to them. These quantities are the temperature, the angular velocities, and the electric potential

$$T = \left(\frac{\partial M}{\partial S} \right)_{J,Q}, \quad \Omega_i = \left(\frac{\partial M}{\partial J_i} \right)_{S,Q}, \quad \Phi = \left(\frac{\partial M}{\partial Q} \right)_{S,J} \qquad (43)$$

It is a matter of straightforward calculation to show that the intensive quantities calculated by Eq. (43) coincide with Eqs. (20), (22), and (26). Thus, these quantities satisfy the first law of thermodynamics

$$dM = TdS + \sum_{i=1}^{k} \Omega_i dJ_i + \Phi dQ. \tag{44}$$

I. CLOSING REMARKS

In this paper, we presented a class of rotating solutions in Gauss-Bonnet gravity in the presence of a nonlinear electromagnetic field. These solutions are not real for the whole spacetime and so, by a suitable transformation, we presented the real solutions. We found that these solutions reduce to the solutions of Gauss–Bonnet–Maxwell gravity as $s = 1$, and reduce to those of Einstein-power Maxwell invariant gravity as α vanishes. Then we studied the kind of singularity and found that, in contrast to the Einstein-power Maxwell invariant gravity, for all values of nonlinear parameter, we can always choose the suitable parameters to have timelike singularity. Also, we found that as in the case of rotating black brane solutions of Einstein-power Maxwell invariant gravity, for $0 < s < \frac{1}{2}$, the asymptotic dominant term of metric function $f(r)$ is charge term, and the presented solutions are not asymptotically AdS, but for the cases $s < 0$ or $s > \frac{1}{2}$, the asymptotic behavior of rotating Einstein-nonlinear Maxwell field solutions are the same as linear AdS case. In the other word, we found that the Gauss-Bonnet does not effect on the asymptotic behavior of the solutions. Then, we applied counterterm method to the solutions with flat boundary at $r =$constant and $t =$ constant, and calculated the finite action and their conserved and thermodynamic quantities. The physical properties of the black brane such as the temperature, the angular velocity, the electric charge and the potential have been computed. We found that the conserved quantities of the black brane do not depend on the Gauss-Bonnet parameter α. Consequently, we obtained the entropy of the black brane through the use of Gibbs-Duhem relation and found that it obeys the area law of entropy. Then, we obtained a Smarr-type formula for the mass as a function of the extensive parameters S , $\mathbf{J}$ and Q, and calculated the intensive parameters T, Ω and Φ. We also showed that the conserved and thermodynamic quantities satisfy the first law of thermodynamics.

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